Journal of Engineering Physics and Thermophysics, Vol. 80, No. 2, 2007

OPTIMUM THICKNESS OF A COOLED WALL SUBJECTED TO LOCAL FRICTIONAL HEATING IN THE REGIME OF SPINNING FRICTION

A. V. Attetkov, I. K. Volkov, and E. S. Tverskaya

UDC 536.2

Using the methods of mathematical simulation, the characteristic features of a stationary temperature field in a cooled plane isotropic wall subjected to local frictional heating from a circular source in the regime of spinning friction are investigated. The possibility of the existence of an optimal wall thickness which ensures a minimum temperature of its most heated point has been established.

In the mathematical theory of heat conduction [1–3] an important role is played by investigations related to the study of the characteristic features of formation of a stationary temperature field in a plane isotropic wall one of the surfaces of which (unprotected) is cooled by the environment and the other (protected or unprotected) is exposed to the action of an external axisymmetrical heat flux, either spatially distributed with a Gaussian-type intensity [4–6] or concentrated (circular, annular) [7–9]. The results of the analysis carried out [4–9] show that irrespective of the type of the acting heat flux there is an optimum thickness of the cooled wall which ensures a minimum steady-state temperature of its most heated point. As far as the present authors are aware, similar results of investigations that establish the possibility of existence of an optimum (in a certain sense) thickness of the cooled wall exposed to a local influence of heat sources of another physical nature, for example, as a result of frictional heating of the wall when the spinning friction regime is implemented [10], are absent, which can be explained by the specifics of the formation of corresponding temperature fields [10, 11]. In what follows, using the conventional terminology [10, 11], the indicated regime of frictional heating will be determined as a particular case of the regime of sliding friction, when relative sliding of contacting bodies represents rotation around the axis coinciding with the common normal to the surface of contact.

In the present work, we consider a plane isotropic wall of constant thickness h, one of the surfaces of which is exposed to the action of the environment with a constant temperature T_{en}^0 and heat-transfer coefficient α^0 and is subjected to local frictional heating in the regime of friction due to the spinning of a circular source of radius R at an angular velocity ω, whereas the other is cooled by the environment with a constant temperature T_{en}^h and heat-transfer coefficient α^h. On the assumption that pressure is distributed uniformly in the zone of frictional contact, the density of the heat flux formed as a result of heat generation in this zone of contact and spent to heat the wall is defined as $q = k\tau\omega r$, $r \le R$, where τ is the specific friction force; $k \cup (0, 1]$ is the coefficient that specifies the distribution of heat fluxes in the materials of the friction pair [12].

The main aim of the investigations carried out is to analyze the characteristic features of a stationary temperature field in a wall being cooled and subjected to local frictional heating in the regime of spinning friction. Based on the above and initial assumptions, we will avail ourselves of the following mathematical model:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\theta}{\partial\rho}\right)+\frac{\partial^2\theta}{\partial x^2}=0\,,\quad \rho\ge 0\,,\quad 0<x<H\,; \tag{1}$$

N. É. Bauman Moscow State Technical University, 5 2nd Baumanovskaya Str., Moscow, 105005, Russia; email: apmath@bmstu.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 80, No. 2, pp. 9–14, March–April, 2007. Original article submitted December 1, 2005.

$$\left.\frac{\partial\theta(\rho, x)}{\partial x}\right|_{x=0} = -Q(\rho) + \mu\mathrm{Bi}\left(\theta(\rho, x)\big|_{x=0} - 1\right); \tag{2}$$

$$\left.\frac{\partial\theta(\rho, x)}{\partial x}\right|_{x=H} = \mathrm{Bi}\left(\theta_{\mathrm{en}} - \theta(\rho, x)\big|_{x=H}\right), \tag{3}$$

where at any fixed $x \cup (0, H)$ the functions $\theta(\rho, x)$ and $Q(\rho)$ as functions of ρ are the inverted transforms of the integral Hankel transformation of zero order [2], which corresponds to the physically evident conditions of symmetry:

$$\lim_{\rho\to+0} \rho\frac{\partial\theta(\rho, x)}{\partial\rho} = 0 = \lim_{\rho\to+\infty} \rho\frac{\partial\theta(\rho, x)}{\partial\rho}; \quad x = \frac{z}{R}; \quad \rho = \frac{r}{R}; \quad \theta = \frac{T}{T_{\mathrm{en}}^0};$$

$$\theta_{\mathrm{en}} = \frac{T_{\mathrm{en}}^h}{T_{\mathrm{en}}^0}; \quad \mathrm{Bi} = \frac{\alpha^h R}{\lambda}; \quad H = \frac{h}{R}; \quad \mu = \frac{\alpha^0}{\alpha^h}; \quad Q = \frac{qR}{\lambda T_{\mathrm{en}}^0}.$$

The form of the function $Q(\rho)$ is established unambiguously by the implemented regime of frictional heat generation. Thus, in particular, in friction due to spinning with a constant angular velocity $\omega \equiv \omega_0$ – const

$$Q(\rho) = Q_0\rho\,[\eta(\rho) - \eta(\rho - 1)].$$

A further analysis is limited by consideration of precisely this regime of local frictional heating of the wall being cooled.

By virtue of the linearity of the initial mathematical model (1)–(3) the function $\theta(\rho, x)$ can be represented as a sum of its three components [7, 8]:

$$\theta(\rho, x) = \theta_1(x) + \theta_2(\rho, x) + \theta_3(\rho, x).$$

The function $\theta_1(x)$ satisfies Eqs. (1)–(3) at $Q(\rho) = 0$, i.e., it is the solution of the problem on finding a stationary temperature field in a plane isotropic wall provided that heat exchange with the environments having constant temperatures T_{en}^0 and T_{en}^h are implemented according to the Newton law [2].

The function $\theta_2(\rho, x)$ satisfies Eq. (1), boundary condition (2) at $\mu = 0$, and a homogeneous analog of boundary condition (3):

$$\left.\frac{\partial\theta_2(\rho, x)}{\partial x}\right|_{x=H} = -\mathrm{Bi}\,\theta_2(\rho, x)\big|_{x=H},$$

i.e., it determines the stationary temperature field in a plane isotropic wall, one of the surfaces of which is cooled by the environment of constant temperature and the other is subjected to a local frictional heating in the regime of spinning friction.

The interpretation of the function $\theta_3(\rho, x)$ that satisfies Eq. (1) and the boundary conditions

$$\left.\frac{\partial\theta_3(\rho, x)}{\partial x}\right|_{x=0} = \mu\,\mathrm{Bi}\left(\theta_2(\rho, x)\big|_{x=0} + \theta_3(\rho, x)\big|_{x=0}\right), \quad \left.\frac{\partial\theta_3(\rho, x)}{\partial x}\right|_{x=H} = -\mathrm{Bi}\,\theta_3(\rho, x)\big|_{x=H},$$

is analogous to the interpretation of the function $\theta_1(x)$.

With account for the analysis carried out and the well-known results obtained in [6–9] we may state that the solution of the problem posed is directly related to the study of the properties of the function $\theta_2(\rho, x)$ at $x = 0$ defined by the following mathematical model:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\theta_2}{\partial\rho}\right)+\frac{\partial^2\theta_2}{\partial x^2}=0\,,\quad \rho\ge 0\,,\quad 0<x<H\,;$$

$$\left.\frac{\partial\theta_2(\rho,x)}{\partial x}\right|_{x=0}=-Q_0\rho\,[\eta(\rho)-\eta(\rho-1)]\,;\quad \left.\frac{\partial\theta_2(\rho,x)}{\partial x}\right|_{x=H}=-\mathrm{Bi}\,\theta_2(\rho,x)\Big|_{x=H}\,. \tag{4}$$

Let

$$V(p,x)=\int\limits_0^{+\infty}\theta_2(\rho,x)\,\rho J_0(p\rho)\,d\rho\,, \tag{5}$$

$$G(p)=\int\limits_0^{+\infty}Q(\rho)\,\rho J_0(p\rho)\,d\rho=Q_0\int\limits_0^{1}\rho^2 J_0(p\rho)\,d\rho=\frac{Q_0}{p}\left\{J_1(p)-\frac{\pi}{2p}\Big[J_1(p)\,\mathrm{H}_0(p)-J_0(p)\,\mathrm{H}_1(p)\Big]\right\} \tag{6}$$

be the images of the integral Hankel transformation of zero order [2] of the functions $\theta_2(\rho, x)$ and $Q(\rho)$, respectively. Using the well-known results of the theory of integral transformations [1–3], we can show that the image $V(p, x)$ is the solutions of the following boundary-value problem:

$$\frac{d^2V}{dx^2}=p^2V\,,\quad 0<x<H\,; \tag{7}$$

$$\left.\frac{dV(p,x)}{dx}\right|_{x=0}=-G(p)\,; \tag{8}$$

$$\left.\frac{dV(p,x)}{dx}\right|_{x=H}=-\mathrm{Bi}\,V(p,x)\Big|_{x=H}\,, \tag{9}$$

where the image $G(p)$ of the function $Q(\rho)$ is determined by equality (6). The solution of Eq. (7) can be presented in the form [7, 8]

$$V(p,x)=c_1(p)\exp(-xp)+c_2(p)\exp(xp)$$

and it must satisfy boundary conditions (8) and (9); this leads to a system of linear algebraic equations for finding the functionals $c_i(p)$, $i \cup \{1, 2\}$:

$$p\Big[c_2(p)-c_1(p)\Big]=-G(p)\,, \tag{10}$$

$$p\left\{c_2(p)\exp(Hp)-c_1(p)\exp(-Hp)\right\}=-\mathrm{Bi}\left\{c_1(p)\exp(-Hp)+c_2(p)\exp(Hp)\right\}. \tag{11}$$

The determination of the functionals $c_i(p)$, $i \cup \{1, 2\}$ that satisfy the system (10)–(11) completes the solution of the problem on finding the stationary temperature field of the cooled wall subjected to local frictional heating in the regime of spinning friction in the images of the integral Hankel transformation (5) employed.

To solve the problem posed it is sufficient to know the temperature $\theta_2(\rho, x)$ of the surface $x = 0$ of the plane isotropic wall considered. In the images of the integral Hankel transformation (5) at $x = 0$ the solution of problem (7)–(9) can be presented in the following form:

$$V(p,0)=\frac{G(p)}{p}\,\frac{1-\dfrac{\mathrm{Bi}-p}{\mathrm{Bi}+p}\exp(-2Hp)}{1+\dfrac{\mathrm{Bi}-p}{\mathrm{Bi}+p}\exp(-2Hp)}\,.$$

Having inverted the integral Hankel transformation of zero order [2], we find a stationary distribution of temperature on the surface of a plane isotropic wall subjected to a local frictional heating:

$$\theta_2(\rho,0)=\int_0^{+\infty}G(p)\,J_0(p\rho)\,\frac{1-\dfrac{\mathrm{Bi}-p}{\mathrm{Bi}+p}\exp(-2Hp)}{1+\dfrac{\mathrm{Bi}-p}{\mathrm{Bi}+p}\exp(-2Hp)}\,dp\,,\quad \rho\ge 0\,, \tag{12}$$

where the image $G(p)$ of the function $Q(\rho)$ is defined by equality (6).

At the limiting values of the wall thicknesses $H = +0$ and $H = +\infty$, the integral on the right side of equality (12) can be represented as [13, 14]

$$\theta_2(\rho,0)\Big|_{H=+0}=\frac{\pi Q_0}{2\mathrm{Bi}}\int_0^{+\infty}p^{-1}\left\{J_0(p)\,\mathrm{H}_1(p)-J_1(p)\,\mathrm{H}_0(p)\right\}J_0(p\rho)\,dp+\frac{Q_0}{\mathrm{Bi}}\begin{cases}1\,, & 0\le\rho<1\,;\\ 0.5\,, & \rho=1\,;\\ 0\,, & \rho>1\,,\end{cases}$$

$$\theta_2(\rho,0)\Big|_{H=+\infty}=Q_0\int_0^{+\infty}p^{-1}J_1(p)\,J_0(p\rho)\,dp-$$

$$-\frac{\pi Q_0}{2}\int_0^{+\infty}p^{-2}\Big[J_1(p)\,\mathrm{H}_0(p)-J_0(p)\,\mathrm{H}_1(p)\Big]J_0(p\rho)\,dp\,,\quad \rho\ge 0\,. \tag{13}$$

Depending on the values of ρ, the first of the integrals on the right side of equality (13) can be calculated explicitly [13, 14]:

$$Q_0\int_0^{+\infty}p^{-1}J_1(p)\,J_0(p\rho)\,dp=\frac{2Q_0}{\pi}\begin{cases}\mathrm{E}(\rho)\,, & 0\le\rho<1\,;\\ 1\,, & \rho=1\,;\\ \rho\left[\mathrm{E}\left(\dfrac{1}{\rho}\right)-\left(1-\dfrac{1}{\rho^2}\right)\mathrm{K}\left(\dfrac{1}{\rho}\right)\right], & \rho>1\,.\end{cases}$$

The results of calculations of a stationary temperature profile of the surface of the wall subjected to local frictional heating in the spinning friction regime are presented partially in Fig. 1. The calculation was performed at Q_0 = 1 and Bi = 1 and at different values of the wall thickness H. We note the principal difference in the forms of the stationary temperature profile given in Fig. 1 and those observed when the cooled wall was heated through a circular region [7–9] and when the wall was exposed to a spatially distributed heat flux with a Gaussian-type intensity [5, 6]. The essential difference is that in the case of local frictional heating of the cooled wall in the spinning friction regime the point of temperature maximum of its surface $x = 0$ depends on the wall thickness. The result obtained is of fundamental importance in determining the conditions of the existence and determination of the optimum thickness of the cooled wall if the optimality criterion is taken to be the condition of ensuring the minimum steady-state temperature at the most heated point of the wall [5–9]. This, in particular, allows one to determine the sufficient conditions for the existence of the optimum thickness of a one-layer wall [5] and of a coated wall [6] exposed to an axisymmetrical heat flux with a Gaussian-type intensity.

In the considered regime of local frictional heating with spinning friction, the maximum heat release is attained on the circle of radius $\rho = 1$, since the linear velocity of the motion of the points of this circle is a maximum.

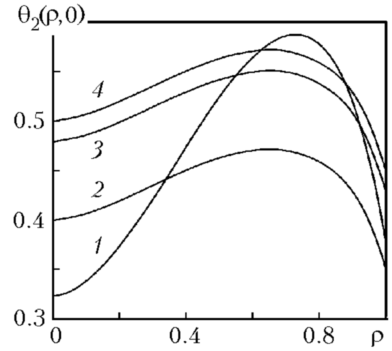

Fig. 1. Stationary temperature profile of the surface x = 0 of the cooled wall of different thicknesses H subjected to local frictional heating in the regime of spinning friction: 1) H = 0.05; 2) 1; 3) 10; 4) +∞.

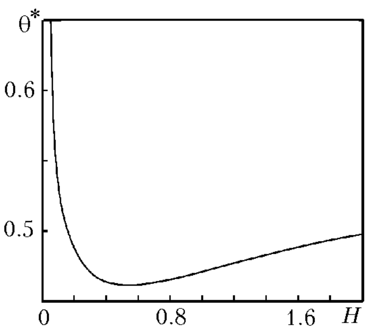

Fig. 2. Dependence of the dimensionless quantity θ^* on the thickness H of the wall subjected to local frictional heating in the regime of spinning friction.

Here it is physically evident that the efflux of heat into the ring $1 < \rho < +\infty$ will prevail over that into the circle $0 \le \rho < 1$. Therefore, the radius ρ^* of the circle formed by the points of the temperature maximum of the surface x = 0 of the cooled wall during its frictional heating in the spinning friction regime must not exceed the radius of the circular source $\rho = 1$, i.e., $\rho^* \cup (0, 1)$. In particular, if we restrict the analysis to the simplest case $H = +\infty$, then according to (13) we obtain

$$\left.\frac{\partial\theta_2\,(\rho, 0)}{\partial\rho}\right|_{H=+\infty} = -\,Q_0 \int\limits_0^{+\infty} J_1\,(p)\,J_1\,(p\rho)\,dp + \frac{\pi Q_0}{2} \int\limits_0^{+\infty} p^{-1} \Big[J_1\,(p)\,\mathrm{H}_0\,(p) - J_0\,(p)\,\mathrm{H}_1\,(p)\Big] J_1\,(p\rho)\,dp\,. \tag{14}$$

Having availed ourselves of equality (14), we can easily see that

$$\exists \lim_{\rho\to 0} \left.\frac{\partial\theta_2(\rho, 0)}{\partial\rho}\right|_{H=+\infty} = 0\,,$$

which allows us to state (see Fig. 1) that (0, 0) is the point of the local minimum. At the same time, according to the results of computational experiment, the temperature maximum at Q_0 = 1 is attained on the circle of radius ρ^* = 0.662.

Figure 2 presents some of the results obtained by calculating the dependence of the dimensionless temperature $\theta_2(\rho^*, 0)$ of the most heated points of the cooled wall subjected to local frictional heating in the regime of spinning friction on its thickness H at Bi = 1. For convenience in the representation of graphical information the wall thickness H is laid on the horizontal axis and the value $\theta^* = \theta_2(\rho^*, 0)/Q_0$ proportional to the steady-state temperature of the points of the temperature maximum of the wall considered is laid on the vertical axis.

The results of computational experiments performed using equalities (12), (6) and partially presented in Fig. 2 point to the possibility of existence of an optimum thickness of the cooled wall in the considered regime of local frictional heating which ensures a minimum steady-state temperature of its most heated points. We also note that an increase in the intensity of heat transfer on the surface $x = H$ of the wall (increase in Bi) leads to a monotonic decrease (down to zero) of an optimum wall thickness.

NOTATION

a, thermal diffusivity, m^2/sec; Bi, Biot number; E(•), complete elliptic integral of the second kind; h, wall thickness, m; H, dimensionless wall thickness; $H_\nu(\bullet)$, Struve function of order ν; $J_\nu(\bullet)$, Bessel function of the first kind of order ν; k, coefficient of the distribution of heat fluxes; K(•), complete elliptic integral of the first kind; p, parameter of the Hankel integral transformation; q, heat flux density, W/m^2; Q, dimensionless heat flux density; r, spatial variable, m; R, radius of a circular source, m; s, parameter of the Laplace integral transformation; T, temperature, K; x, dimensionless spatial variable; z, spatial variable, m; α, heat-transfer coefficient, $W/(m^2 \cdot K)$; $\eta(\bullet)$, Heaviside function; θ, dimensionless temperature; λ, thermal conductivity, $W/(m \cdot K)$; μ, similarity simplex as a measure of the relationship between the coefficients of heat transfer on the surfaces $x = 0$ and $x = h$ of the wall; ρ, dimensionless spatial variable; τ, specific friction force; N/m^2; ω, angular velocity of rotation 1/sec. Subscripts: 0 and h, surfaces $x = 0$ and $x = h$ of the wall, respectively; en, environment.

REFERENCES

1. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* [Russian translation], Nauka, Moscow (1964).
2. A. V. Luikov, *Heat Conduction Theory* [in Russian], Vysshaya Shkola, Moscow (1967).
3. É. M. Kartashov, *Analytical Methods in the Theory of the Thermal Conductivity of Solids* [in Russian], Vysshaya Shkola, Moscow (2001).
4. V. S. Zarubin, *Calculation and Optimization of Thermal Insulation* [in Russian], Énergoatomizdat, Moscow (1991).
5. V. S. Zarubin, An optimal thickness of a cooled wall subjected to local heating, *Izv. Vyssh. Uchebn. Zaved., Mashinostroenie*, No. 10, 18–21 (1970).
6. A. V. Attetkov, I. K. Volkov, and E. S. Tverskaya, The optimum thickness of a cooled coated wall exposed to local pulse-periodic heating, *Inzh.-Fiz. Zh.*, **74**, No. 6, 82–87 (2001).
7. A. V. Attetkov and E. S. Tverskaya, Characteristic features of formation of a stationary temperature field in a cooled wall subjected to pulse-periodic heating through a circular region, in: *Present-day Natural-Scientific and Humanitarian Problems* [in Russian], Logos, Moscow (2005), pp. 413–423.
8. A. V. Attetkov, I. K. Volkov, and E. S. Tverskaya, Optimum thickness of a cooled wall in local pulse-periodic heating, *Inzh.-Fiz. Zh.*, **78**, No. 2, 16–23 (2005).
9. A. V. Attetkov, I. K. Volkov, and E. S. Tverskaya, Optimal thickness of a cooled coated wall subjected to local pulse-periodic heating, *Teplofiz. Vys. Temp.*, **43**, No. 3, 466–473 (2005).
10. M. V. Korovchinskii, Principles of the theory of thermal contact in local friction, in: *Friction and Lubrication Problems* [in Russian], Nauka, Moscow (1968), pp. 5–72.
11. A. A. Evtushenko and E. G. Ivanik, Thermostressed state at a local thermal contact in turning friction, *Inzh.-Fiz. Zh.*, **69**, No. 1, 72–78 (1996).
12. O. V. Pereverzeva and V. A. Balakin, Distribution of heat between bodies in friction, *Trenie Iznos*, **13**, No. 3, 507–516 (1992).
13. I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products* [in Russian], Nauka, Moscow (1971).
14. H. Bateman and A. Erdelyi, *Higher Transcendental Functions. Bessel Funcitons, Parabolic Cylinder Functions, Orthogonal Polynomials* [Russian translation], Nauka, Moscow (1966).